

Conformal covariance for the powers of the Dirac operator

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Abstract

A new proof of the conformal covariance of the powers of the flat Dirac operator is obtained. The proof uses their relation with the Knapp-Stein intertwining operators for the spinorial principal series. We also treat the compact picture, i.e. the corresponding operators on the sphere, where certain polynomials of the Dirac operator appear. This gives a new representation-theoretic framework for earlier results in [4, 8, 12].¹

1 Introduction

The powers of the (flat) Dirac operator are known to satisfy a covariance property with respect to the Möbius group (see [2, 14, 8]). We give a new proof by interpreting the powers of the Dirac operator as *residues* of a meromorphic family of Knapp-Stein intertwining operators. The proof is elementary and does not involve any Clifford analysis. We also give in the last section the corresponding meromorphic family and its residues on the sphere. This corresponds to the so-called *compact picture* of the induced representations. In particular we find as residues on the sphere the polynomials $D(D^2 - 1^2) \cdots (D^2 - m^2)$ of the Dirac operator D . These were found earlier by other methods in [8, 12] (in these references the polynomials are not with constant coefficients, and it still needs some consideration to see that they may be identified with the polynomials above).

2 The conformal group of the sphere

Let $(E^{n+1}, (\cdot, \cdot))$ be a Euclidean vector space of dimension $n+1$, and denote by (x, y) the inner product of two vectors x and y . Let $S = S^n$ be the unit

¹2000 Mathematics Subject Classification : 22E45, 43A80

sphere of E . The *Lorentz space* is

$$E^{1,n+1} = \mathbb{R} \oplus E = \{(t, x), t \in \mathbb{R}, x \in E^{n+1}\}$$

endowed with the symmetric bilinear form $[\cdot, \cdot]$ given by

$$[(t, x), (u, y)] = tu - (x, y) .$$

Let \mathcal{S} be the set of isotropic lines in $E^{1,n+1}$. The map

$$S \ni x \longmapsto d_x = \mathbb{R}(1, x) \in \mathcal{S}$$

is a one-to-one correspondance of S onto \mathcal{S} , which is moreover a diffeomorphism for the canonical differential structures on S (as a submanifold of E^{n+1}) and \mathcal{S} (as a submanifold of the projective space of $E^{1,n+1}$).

Let $G = SO_0(E^{1,n+1}) \simeq SO_0(1, n+1)$ be the connected component of the neutral element in the group of isometries of $E^{1,n+1}$. Then G acts on \mathcal{S} and hence on S . This action is *conformal* in the sense that for any g in G and $x \in S$, the differential $Dg(x) : T_x \longrightarrow T_{g(x)}$ is a similarity of the tangent space T_x of S at x , i.e.

$$Dg(x) = \kappa(g, x) r(g, x) ,$$

where $r(g, x)$ is a positive isometry of T_x into $T_{g(x)}$ and $\kappa(g, x) > 0$ is the *conformal factor* of g at x .

The group $K = SO(E^{n+1}) \simeq SO(n+1)$ is a maximal compact subgroup of G , associated to the standard Cartan involution $g \longmapsto \theta(g) = (g^t)^{-1}$. The group K acts transitively on S .

Let (e_0, \dots, e_n) be an orthonormal basis of E^{n+1} , and choose e_0 as origin in S . Let $E^n = e_0^\perp$ be the hyperplane orthogonal to e_0 in E^{n+1} . The stabilizer of e_0 in K is the subgroup $M \simeq SO(n)$, and this gives a realization of $S \simeq K/M$ as a *compact Riemannian symmetric space*. On the other hand, let P be the stabilizer of e_0 in G . Then P is a parabolic subgroup of G , and $S \simeq G/P$ is a realization of S as a *flag manifold*.

Let $e_{-1} = (1, 0, \dots, 0) \in E^{1,n+1}$, so that $\{e_{-1}, e_0, \dots, e_n\}$ is a basis of $E^{1,n+1}$. Introduce the subgroups of G defined by

$$A = \left\{ a_s = \begin{pmatrix} \cosh s & \sinh s & 0 & \dots & 0 \\ \sinh s & \cosh s & 0 & \dots & 0 \\ 0 & 0 & 1 & & \\ \vdots & \vdots & & \ddots & \\ 0 & 0 & & & 1 \end{pmatrix}, \quad s \in \mathbb{R} \right\}$$

and

$$N = \left\{ n_u = \begin{pmatrix} 1 + \frac{|u|^2}{2} & -\frac{|u|^2}{2} & & u^t \\ \frac{|u|^2}{2} & 1 - \frac{|u|^2}{2} & & u^t \\ & & 1 & \\ u & -u & \ddots & \\ & & & 1 \end{pmatrix}, \quad u \in \mathbb{R}^n \simeq E^n \right\}.$$

Then $P = MAN$ is a Langlands decomposition of P . Let $\overline{N} = \theta N$ be the opposite unipotent subgroup, given by

$$\overline{N} = \left\{ \overline{n}_v = \begin{pmatrix} 1 + \frac{|v|^2}{2} & \frac{|v|^2}{2} & & v^t \\ -\frac{|v|^2}{2} & 1 - \frac{|v|^2}{2} & & -v^t \\ & & 1 & \\ v & v & \ddots & \\ & & & 1 \end{pmatrix}, \quad v \in \mathbb{R}^n \right\}.$$

We also let

$$w = \begin{pmatrix} 1 & & & & \\ & -1 & & & \\ & & -1 & & \\ & & & 1 & \\ & & & & \ddots \\ & & & & & 1 \end{pmatrix}.$$

The element w is in K , acts on S^n by $(x_0, x_1, x_2, \dots, x_n) \mapsto (-x_0, -x_1, x_2, \dots, x_n)$, satisfies $wa_s w^{-1} = a_{-s}$, thus realizes the non trivial Weyl group element.

The map

$$c : v \mapsto \left(\frac{|v|^2 - 1}{|v|^2 + 1}, \frac{2}{|v|^2 + 1} v \right)$$

is a diffeomorphism from E^n onto $S \setminus \{e_0\}$. Its inverse is the classical *stereographic projection* from the source e_0 .

As a convention let $\overline{E}^n = E^n \cup \infty$ be the one-point compactification of E^n . Then clearly the map c can be extended to \overline{E}^n by setting $c(\infty) = e_0$, to get a one-to-one correspondance between \overline{E}^n and S^n . This allows to transfer the action of G on S to an action of G on \overline{E}^n . In this way, the group G is realized as a group of rational conformal transformations of E^n , usually called the *Möbius group* $M_+(\overline{E}^n)$. The subgroup P is now realized by affine similarities, the group M is the group of rotations of E with center at 0,

A is the group of positive dilations with center 0 and N is the group of translations of E . The element w acts as the *twisted inversion*

$$(x_1, x_2, \dots, x_n) \mapsto \left(-\frac{x_1}{|x|^2}, \frac{x_2}{|x|^2}, \dots, \frac{x_n}{|x|^2} \right).$$

When using this model for the sphere, we refer to the *noncompact picture*.

3 The Vahlen-Maass-Ahlfors realization of the twofold covering of the Möbius group

There is a quite useful realization of (a twofold covering of) G acting on \overline{E}_n via the Clifford algebra Cl_{n-1} , initiated by Vahlen ([17]), revived by Maass ([13]) and well presented by Ahlfors ([1], see also [10], [15]).

Let E^{n-1} be the vector subspace generated by e_2, \dots, e_n , and form the *Clifford algebra* $Cl_{n-1} = Cl(E^{n-1})$, i.e. the algebra (with unit 1) generated by the vector space E^{n-1} and the relations

$$uv + vu + 2(u, v) = 0.$$

The space E^n is identified with the subspace $\mathbb{R} \oplus E^{n-1}$ of $Cl(E^{n-1})$, the element e_1 corresponding to the unit of the Clifford algebra. Following Ahlfors, elements of E^n will be called *vectors*.

Recall the three involutions of the Clifford algebra : the *principal automorphism* $a \mapsto a'$, obtained by sending e_j to $-e_j$ for $2 \leq j \leq n$, the *reversion* $a \mapsto a^*$ which is the antiisomorphism mapping e_j to e_j for $2 \leq j \leq n$ and the *conjugation* $a \mapsto \bar{a}$ which is the composition of the two first.

There exists a canonical inner product on Cl_{n-1} extending the inner product on E^{n-1} , the corresponding norm is denoted by $|\cdot|$.

For vectors $x = x^*$, $x' = \bar{x}$, and $x\bar{x} = x_1^2 + \dots + x_n^2 = |x|^2$, so that any vector $x \neq 0$ is invertible with inverse equal to $x^{-1} = \frac{\bar{x}}{|x|^2}$.

The *Clifford group* Γ_n is the set of all elements of the Clifford algebra Cl_{n-1} which can be written as products on non-zero vectors. If a, b are in Γ_n , then

$$a\bar{a} = |a|^2, \quad |ab| = |a||b|$$

Let a be in Γ_n . Then a is invertible and $a^{-1} = \frac{\bar{a}}{|a|^2}$. For x in E^n , $axa'^{-1} \in E^n$, and the map $x \mapsto axa'^{-1}$ is a positive isometry of E^n .

Definition 3.1. A matrix $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a Clifford matrix if

- i) $a, b, c, d \in \Gamma_n \cup \{0\}$
- ii) $ad^* - bc^* = 1$
- iii) ab^* and $cd^* \in E^n$

Theorem 3.1. *Under matrix multiplication, the Clifford matrices form a group, denoted by $SL_2(\Gamma_n)$.*

The following observation will be useful later : for $a, b \in \Gamma_n$, the conditions $ab^* \in E^n$ and $a^{-1}b \in E^n$ are equivalent.

Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $SL_2(\Gamma_n)$ and x in \overline{E}^n . We recall the meaning of the expression $g(x) = (ax + b)(cx + d)^{-1}$.

First $cx + d$ is either 0 or is invertible :

- if $c \neq 0$, c is in Γ_n . Condition iii) and the remark imply that $c^{-1}d \in E^n$, so that $cx + d = c(x + c^{-1}d)$ is either 0 or invertible.
- if $c = 0$, then ii) implies that $d \neq 0$ so that $cx + d = d$ is in Γ_n , hence is invertible.

Next, observe that $ax + b$ and $cx + d$ cannot be both 0. In fact assume the contrary. By ii), a and c can not be both 0. So, assume that $a \neq 0$. Hence a is invertible and $x = -a^{-1}b = b^*a^{*-1}$. Hence

$$0 = cx + d = -cb^*(a^*)^{-1} + d .$$

Multiplying both sides on the left by a^* yields $-c^*b + da^* = 0$ which is incompatible with ii). A similar argument holds if $c \neq 0$.

So, if $cx + d \neq 0$, $(ax + b)(cx + d)^{-1}$ is a well defined element of Cl_{n-1} . If $cx + d = 0$, then set $(ax + b)(cx + d)^{-1} = \infty$. Finally, let $(a\infty + b)(c\infty + d)^{-1} = ac^{-1}$ is $c \neq 0$ and $= \infty$ if $c = 0$.

Theorem 3.2.

i) For any g in $SL_2(\Gamma_n)$ and $x \in \overline{E}^n$, $g(x) = (ax + b)(cx + d)^{-1}$ belongs to \overline{E}^n

ii) The map $\iota_g : \overline{E}^n \ni x \mapsto g(x) \in \overline{E}^n$ belongs to the Möbius group $M(\overline{E}^n)$

iii) The homomorphism $g \mapsto \iota_g$ is a twofold covering of the Möbius group, with kernel $\pm \text{Id}_2$.

See [1] for a proof.

In the sequel , we let $\tilde{G} = SL_2(\Gamma_n)$. The stabilizer of ∞ in \tilde{G} is the subgroup

$$\tilde{P} = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}, \quad ad^* = 1, \quad ab^* \in E^n \right\} .$$

The Langlands decomposition of \tilde{P} is $\tilde{P} = \tilde{L}N = \widetilde{M}AN$, where

$$\begin{aligned} N &= \left\{ \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix}, \quad v \in E^n \right\} \\ \tilde{L} &= \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{*-1} \end{pmatrix}, \quad a \in \Gamma^n \right\} \\ A &= \left\{ a_t = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}, \quad t > 0 \right\} \end{aligned}$$

and the group \widetilde{M} is realized as

$$\widetilde{M} = \left\{ \begin{pmatrix} m & 0 \\ 0 & m^{*-1} \end{pmatrix}, \quad m \in \Gamma_n, |m| = 1 \right\}.$$

The element $\begin{pmatrix} m & 0 \\ 0 & m^{*-1} \end{pmatrix}$ of \widetilde{M} acts on E^n by $\sigma_m : x \mapsto mxm^*$ and the map $\sigma : m \mapsto \sigma_m$ is a twofold covering of $\widetilde{M} \simeq Spin_n$ onto $SO(n)$.

Let $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, so that, for $t > 0$ $a_t = \exp \log t H$. A complex linear form on \mathfrak{a} is identified with the complex number λ equal to the value of the linear form on the element H . The half-sum of the roots ρ corresponds to the number n , and we let $a_t^\lambda = t^\lambda, t > 0, \lambda \in \mathbb{C}$.

The group N acts by translations $x \mapsto x + v$.

An element of \overline{N} is of the form $\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}$ where $x \in E^n$. We will identify \overline{N} with E_n .

The non trivial Weyl group element is realized by the matrix $w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

The *Bruhat decomposition* of an element $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where $a \neq 0$ is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ ca^{-1} & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & a^{*-1} \end{pmatrix} \begin{pmatrix} 1 & a^{-1}b \\ 0 & 1 \end{pmatrix}.$$

The proof of this identity reduces to $ca^{-1}b + a^{*-1} = d$, which is a consequence of the assumptions on a, b, c and d (see Definition 3.1).

Proposition 3.1. *Let $g \in \tilde{G}$, $x, y \in E^n$ and assume that $g(x), g(y) \in E^n$. Then*

$$g(x) - g(y) = (cy + d)^{* -1} (x - y)(cx + d)^{-1} . \quad (1)$$

Proposition 3.2. *Let g be in \tilde{G} , $x \in E^n$ and assume that $g(x) \in E^n$. Then the differential of the action of g at x is given by*

$$Dg(x)\xi = (cx + d)^{* -1} \xi (cx + d)^{-1} . \quad (2)$$

The conformal factor of g at x is given by $\kappa(g, x) = |cx + d|^{-2}$ and the rotation factor is given by $\sigma((cx + d)' |cx + d|^{-1})$.

Proposition 1 is proved in [1] (see also [10]). The formula for the differential in Proposition 3.2 is a consequence. For the last part of the proposition, observe that $a = (cx + d)$ is in Γ_n and

$$a^{* -1} \xi a^{-1} = |a|^{-2} \left(\frac{a}{|a|} \right)^{* -1} \xi \left(\frac{a}{|a|} \right)^{-1} = |a|^{-2} \frac{a'}{|a|} \xi \left(\frac{a'}{|a|} \right)^* .$$

Another formula will be useful later.

Proposition 3.3. *Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be in $SL_2(\Gamma_n)$. Let $x \in E^n$, assume that g is defined at x and let $y = (ax + b)(cx + d)^{-1}$. Then*

$$(-c^* y + a^*) = (cx + d)^{-1} \quad (3)$$

Proof. The identity follows from

$$(-c^* y + a^*)(cx + d) = -c^*(ax + b) + a^*(cx + d) = -c^*b + a^*d ,$$

and the fact that $a^*d - c^*b = 1$, a consequence of the conditions satisfied by a, b, c, d (see Definition 3.1). \square

The map $\tilde{\theta}$ defined on \tilde{G} by

$$\tilde{\theta} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d' & -c' \\ -b' & a' \end{pmatrix}$$

is an involution of \tilde{G} , which covers the standard involution θ on G . The fixed points set of $\tilde{\theta}$ is the subgroup \tilde{K} given by

$$\tilde{K} = \left\{ \begin{pmatrix} a & b \\ -b' & a' \end{pmatrix}, a, b, \in \Gamma_n \cup \{0\}, |a|^2 + |b|^2 = 1, ab^* \in E^n \right\} .$$

The subgroup \tilde{K} is a maximal compact subgroup of \tilde{G} , isomorphic to $Spin(n+1)$ and is a twofold covering of K .

The sphere S^n can be interpreted as \tilde{G}/\tilde{P} (flag manifold) and as \tilde{K}/\tilde{M} (compact symmetric space).

To determine the Lie algebra of $Spin_n$ in this model, we will describe one-parameter groups, and find the associated vector field.

First consider, for $2 \leq j \leq r$

$$m_t = \cos \frac{t}{2} + \sin \frac{t}{2} e_j, \quad t \in \mathbb{R} .$$

Then

$$\sigma_{m_t} = \begin{pmatrix} \cos t & -\sin t & & \\ & 1 & & \\ \sin t & & \cos t & \\ & & & 1 \end{pmatrix}$$

For $2 \leq j < k$, let

$$m_t = \cos \frac{t}{2} e_j + \sin \frac{t}{2} e_k .$$

Then

$$\sigma_{m_t} = \begin{pmatrix} 1 & & & \\ & \cos t & -\sin t & \\ & & 1 & \\ & \sin t & & \cos t \\ & & & & 1 \end{pmatrix}$$

So, a basis of the Lie algebra of the spin group is

$$\frac{1}{2} e_j, \quad 2 \leq j \leq n, \quad \frac{1}{2} e_j e_k, \quad 2 \leq j < k \leq n .$$

4 The spinor representation

We recall some well-known results on the spinor representations. We will use the standard realization of $Spin$ in Cl_n , (not to be confused with the realization of the same group in Cl_{n-1} we used in the previous section), namely

$$Spin_n = \{a = v_1 \dots v_{2k}, \quad v_j \in \mathbb{R} \oplus \mathbb{R}^n, |v_j| = 1, k \in \mathbb{N}\} .$$

A finite dimensional complex Hilbert space \mathcal{H} is said to be a *Clifford module* if there exists n skew-Hermitian operators E_1, \dots, E_n on \mathcal{H} , such that

$$E_i E_j + E_j E_i = -2\delta_{ij} \text{Id}, \quad 1 \leq i, j \leq n. \quad (4)$$

By the universal property of the Clifford algebra, there exists a (uniquely defined) representation (τ, \mathcal{H}) of the *complex* Clifford algebra $\mathbb{C}l_n$, which satisfies $\tau(e_j) = E_j$, and conversely, any representation of the Clifford algebra is obtained in this manner. Note that, for any $a \in \mathbb{C}l_n$, $\tau(\bar{a})$ is the adjoint of $\tau(a)$ for the Hilbert product on Σ .

Viewing $Spin_n$ as a subset of $\mathbb{C}l_n$, we obtain by restriction of τ a representation of $Spin_n$ on \mathcal{H} , still denoted by τ , which is unitary by the last remark.

The results concerning the *irreducible* representations of $\mathbb{C}l_n$ depend on the parity of n .

Assume first that n is *even*. Then the *complex Clifford algebra* $\mathbb{C}l_n$ has a unique irreducible representation (up to isomorphism) for n even. Let Σ_n the complex (finite-dimensional) Hilbert space on which the representation acts, and denote by $\sigma : \mathbb{C}l_n \rightarrow \text{End}(\Sigma_n)$ the representation. When restricted to $Spin_n$ (or equivalently to the even part $\mathbb{C}l_n^{ev}$), the representation (σ, Σ_n) splits into two inequivalent representations.

In case n is *odd*, then there are two inequivalent irreducible representations, denoted by (σ^+, Σ_n^+) and (σ^-, Σ_n^-) . To distinguish them, let

$$\omega^{\mathbb{C}} = i^{\lfloor \frac{n+1}{2} \rfloor} e_1 e_2 \dots e_n$$

be the *volume element*. Observe that the oddness of n implies that $\omega^{\mathbb{C}}$ is in the center of $\mathbb{C}l_n$. By Schur's lemma, $\tau(\omega) = \pm \text{Id}$ for τ an irreducible representation. Hence $\sigma^{\pm}(\omega^{\mathbb{C}}) = \pm \text{Id}$ on Σ_n^{\pm} , which distinguishes the two representations, and shows that they are not equivalent. When restricted to $\mathbb{C}l_n^{ev}$ (or to $Spin_n$), both restrictions of σ^{\pm} stay irreducible and are equivalent representations. Let $\Sigma_n = \Sigma_n^+ \oplus \Sigma_n^-$, and let $\sigma = \sigma^+ \oplus \sigma^-$.

In any case of parity, we call (σ, Σ_n) the *spinor representation* of $\mathbb{C}l_n$ (or of $Spin_n$). On Σ , there is a Hermitian scalar product for which $\sigma(x)$ is unitary for any $x \in \mathbb{R}^n$ with unit length. For this inner product, Σ is a Clifford module. For $1 \leq j \leq n$, let $E_j = \sigma(e_j)$. Then the E_j 's are skew Hermitian and satisfy the defining relations (4).

Finally, we have to connect the standard realization of $Spin_n$ with the realization of \widetilde{M} in the Vahlen-Maass-Ahlfors approach. The linear map $\gamma : E^{n-1} \mapsto \mathbb{C}l_n^{ev}$ given by

$$\gamma(e_j) = e_1 e_j, \quad 2 \leq j \leq n$$

satisfies $\gamma(e_i)\gamma(e_j) + \gamma(e_j)\gamma(e_i) = -2\delta_{ij}$ and hence can be extended to yield an isomorphism (still denoted by γ) of $Cl(E^{n-1})$ onto Cl_n^{ev} . The map γ induces an isomorphism of \widetilde{M} onto $Spin_n$. For $a \in Cl(E^{n-1})$, let $\tau(a) = \sigma(\gamma(a))$. Then (τ, Σ) is a representation of $Cl(E^{n-1})$ and, by restriction to \widetilde{M} , a representation of \widetilde{M} , called the *spinor representation*. For $x \in E^n$,

$$\tau(x_1 + x_2 e_2 + \cdots + x_n e_n) = x_1 \text{Id} + x_2 E_1 E_2 + \cdots + x_n E_1 E_n . \quad (5)$$

Recall that the element $\tilde{w} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is a representative of the nontrivial Weyl group element.

Proposition 4.1. *Let τ be the spinor representation of \widetilde{M} , and let $w\tau$ be the representation of \widetilde{M} given by $w\tau(m) = \tau(\tilde{w}^{-1}m\tilde{w})$. Then*

$$w\tau = \tau' ,$$

where τ' is the restriction to \widetilde{M} of the representation of $\mathbb{C}l_{n-1}$ given by $\tau'(a) = \tau(a')$, for $a \in \mathbb{C}l_{n-1}$.

Proof. For $a \in \Gamma_n$,

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & a^{*-1} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} a^{*-1} & 0 \\ 0 & a \end{pmatrix} .$$

If moreover, $|a| = 1$, then $a^{*-1} = a'$, so that the automorphism of \widetilde{M} induced by \tilde{w} coincides with the principal automorphism. The statement follows. \square

Proposition 4.2. *For any $a \in Cl(E^{n-1})$,*

$$E_1 \tau'(a) = \tau(a) E_1 . \quad (6)$$

Proof. It suffices to verify the statement for any vector $x \in E^n$. Let $x = x_1 + x_2 e_2 + \cdots + x_n e_n$. Then

$$\tau(x) E_1 = (x_1 \text{Id} + x_2 E_1 E_2 + \cdots + x_n E_1 E_n) E_1 = x_1 E_1 + x_2 E_2 + \cdots + x_n E_n$$

whereas

$$E_1 \tau'(x) = E_1 (x_1 \text{Id} - x_2 E_1 E_2 - \cdots - x_n E_1 E_n) = x_1 E_1 + x_2 E_2 + \cdots + x_n E_n .$$

\square

5 The principal spinorial series of \tilde{G} and the associated intertwining operators

Let us first recall the general theory of Knapp-Stein intertwining operators. Let G be a semisimple Lie group (connected and with finite center), P a minimal parabolic subgroup. Let θ be a Cartan involution, with fixed points K , which is a maximal compact subgroup of G . Let $P = MAN$ be a Langlands decomposition of P adapted to θ . In particular, $M = P \cap K$. Let M' be the normalizer of A and $W \simeq M'/M$ be the corresponding Weyl group. Let $X = G/P \simeq K/M$, and let $o = eP$ be the origin in X . Let \mathfrak{a} be the Lie algebra of A , and let \exp be the exponential map from \mathfrak{a} onto A . Let $\rho \in \mathfrak{a}'$ (the dual of \mathfrak{a}) be the half-sum of the positive roots relative to N .

The map

$$K \times A \times N \ni (k, a, n) \longmapsto kan \in G$$

is a diffeomorphism onto G . If g is in G , we write $g = \kappa(a) \exp H(g) \nu(g)$ for the *Iwasawa decomposition*.

Let $\overline{N} = \theta N$. The map

$$\overline{N} \times M \times A \times N \ni (\overline{n}, m, a, n) \longmapsto \overline{n}man \in G$$

is a diffeomorphism onto a dense open set of G . For g an element in the image, let

$$g = \overline{n}(g)m(g)a(g)n(g)$$

be the corresponding *Bruhat decomposition*.

Let τ be a unitary representation of M on a (finite dimensional) Hilbert space V and let λ be a complex linear form on $\mathfrak{a} = \text{Lie}(A)$. Let τ_λ be the representation of P defined by

$$\tau_\lambda(man) = a^\lambda \tau(m), \quad m \in M, a \in A, n \in N,$$

where $a^\lambda = e^{\lambda(\log a)}$.

Form the *induced representation* $\pi_{\tau, \lambda} = \text{Ind}_{MAN}^G(\sigma \otimes \exp \lambda \otimes 1)$. We will work with the *noncompact picture* of this induced representation. Introduce the space $L_\lambda^2(\overline{N})$ as the space of functions f on \overline{N} , valued in V , which satisfy

$$\int_{\overline{N}} |f(x)|^2 e^{2\Re \lambda(H(\overline{n}))} d\overline{n} < +\infty.$$

We state the noncompact realization of the induced representation as a proposition (see [11] ch VII).

Proposition 5.1. *For $g \in G$,*

$$\pi_{\tau,\lambda}(g)f(\bar{n}) = e^{-(\lambda+\rho)\log a(g^{-1}\bar{n})}\tau(m(g^{-1}\bar{n}))^{-1}f(\bar{n}(g^{-1}\bar{n})) . \quad (7)$$

defines a representation $\pi_{\tau,\lambda}$ of G by bounded operators on $L^2_\lambda(\bar{N})$.

Let w be an element of W , and choose a representative (still denoted by w) in M' . Let $w\tau$ be the representation of \widetilde{M} defined by $w\tau(m) = \tau(\widetilde{w}^{-1}m\widetilde{w})$. The Knapp-Stein theory of intertwining operators offers a construction of an intertwining operator between $\pi_{\tau,\lambda}$ and $\pi_{w\tau,w\lambda}$. We state it as a proposition (see [11] ch VII, (7.39).

Set, for f a function on \bar{N} and $x \in \bar{N}$

$$J_{\tau,\lambda,w}f(x) = \int_{\bar{N}} e^{(-\rho+\lambda)\log a(w^{-1}\bar{n})}\tau(m(w^{-1}\bar{n}))f(x\bar{n})d\bar{n}$$

Proposition 5.2. *The operator J_λ is an intertwining operator between $\pi_{\tau,\lambda}$ and $\pi_{w\tau,w\lambda}$, namely for any $g \in G$,*

$$J_{\tau,\lambda,w} \circ \pi_{\tau,\lambda}(g) = \pi_{w\tau,w\lambda}(g) \circ J_{\tau,\lambda,w} .$$

The proposition lets aside the convergence of the integral, which is true for λ in some open set of $\mathfrak{a}'_{\mathbb{C}}$. The intertwining operator $J_{\tau,\lambda,w}$ can then be extended meromorphically to the whole space.

This general scheme applies to our situation. Let (τ, Σ) be the spinor representation of \widetilde{M} and let $\lambda \in \mathbb{C}$. Define the representation τ_λ of \widetilde{P} by

$$\tau_\lambda(ma_t n) = t^{2\lambda}\tau(m) .$$

Let $\pi_\lambda = \text{Ind}_{\widetilde{P}}^{\widetilde{G}} \tau_\lambda$. Following the procedure just described above, we obtain the following realization of these representations (*noncompact picture*).

Theorem 5.1. *For $\lambda \in \mathbb{C}$ and $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the formula*

$$\pi_\lambda(g)f(x) = |d^* - b^*x|^{-2\lambda-n}\tau\left(\frac{d^* - b^*x}{|d^* - b^*x|}\right)^{-1}f((-c^* + a^*x)(d^* - b^*x)^{-1})$$

defines a representation of \widetilde{G} by bounded operators on $L^2_\lambda(E^n, \Sigma)$.

Proof. For $x \in E$,

$$g^{-1}\bar{n}_x = \begin{pmatrix} d^* - b^*x & -b^* \\ -c^* + a^*x & a^* \end{pmatrix} .$$

The components in the Bruhat decomposition of this element are

$$(ma)(g^{-1}\bar{n}_x) = d^* - b^*x, \quad \bar{n}(g^{-1}\bar{n}_x) = (-c^* + a^*x)(d^* - b^*x)^{-1}.$$

Hence the formula is a consequence of the general statement. \square

There is another closely related representation. In fact, the same construction can be done using the representation τ' of \widetilde{M} we introduced earlier instead of τ . The corresponding representation will be denoted by π'_λ . It is related to π_λ by the following elementary result, which follows from Proposition 6.

Proposition 5.3. *For any $g \in \widetilde{G}$*

$$E_1 \pi'_\lambda(g) = \pi_\lambda(g) E_1. \quad (8)$$

We now apply Knapp-Stein theory to get an intertwining operator between π_λ and $\pi'_{-\lambda}$.

Let $\bar{n} = \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix}$. Then

$$w^{-1}\bar{n} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} = \begin{pmatrix} y & 1 \\ -1 & 0 \end{pmatrix}$$

so that

$$t(w^{-1}\bar{n}) = |y|, \quad m(w^{-1}\bar{n}) = \frac{y}{|y|}.$$

Hence the Knapp-Stein operator is given by

$$J_\lambda f(x) = \int_{E^n} |y|^{2\lambda-n} \tau\left(\frac{y}{|y|}\right) f(x-y) dy$$

Theorem 5.2. *For $\lambda \in \mathbb{C}$, let*

$$J_\lambda f(x) = \int_{E^n} |y|^{2\lambda-n} \tau\left(\frac{y}{|y|}\right) f(x-y) dy. \quad (9)$$

For $\Re \lambda > 0$ and $f \in L^2_\lambda(E^n, \Sigma)$, the integral converges and the operator J_λ thus defined is bounded on $L^2_\lambda(E^n, \Sigma)$ and for any $g \in \widetilde{G}$ satisfies

$$J_\lambda \pi_\lambda(g) = \pi'_{-\lambda}(g) J_\lambda. \quad (10)$$

Although this is a consequence of the Knapp-Stein theory, we may offer a direct proof (compare with [10]).

Proof. Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Let $\gamma = \begin{pmatrix} a^* & -c^* \\ -b^* & d^* \end{pmatrix}$ which is easily seen to be an element of \tilde{G} .

Let f be in $\mathcal{C}_c^\infty(E^n, \Sigma)$. Then

$$J_\lambda f(x) = \int_{E^n} |x - y|^{2\lambda-n} \tau\left(\frac{x-y}{|x-y|}\right) f(y) dy ,$$

so that

$$\begin{aligned} \pi'_{-\lambda}(g) J_\lambda f(x) &= \dots \\ &= |d^* - b^* x|^{2\lambda-n} \tau'\left(\frac{d^* - b^* x}{|d^* - b^* x|}\right)^{-1} \int_{E^n} |\gamma(x) - y|^{2\lambda-n} \tau\left(\frac{\gamma(x) - y}{|\gamma(x) - y|}\right) f(y) dy . \end{aligned}$$

Use the change of variable $y = \gamma(z)$, and hence $dy = |d^* - b^* z|^{-2n} dz$ to get

$$|d^* - b^* x|^{-2\lambda-n} \tau'\left(\frac{d^* - b^* x}{|d^* - b^* x|}\right)^{-1} \int_{E^n} |\gamma(x) - \gamma(z)|^{2\lambda-n} \tau\left(\frac{\gamma(x) - \gamma(z)}{|\gamma(x) - \gamma(z)|}\right) |d^* - b^* z|^{-2n} f(\gamma(z)) dz .$$

Now, using (1)

$$\begin{aligned} \gamma(x) - \gamma(z) &= (d^* - b^* x)^{* -1} (x - z) (d^* - b^* z)^{-1} \\ |\gamma(x) - \gamma(z)| &= |d^* - b^* x|^{-1} |x - z| |d^* - b^* z|^{-1} \\ \frac{\gamma(x) - \gamma(z)}{|\gamma(x) - \gamma(z)|} &= \left(\frac{d^* - b^* x}{|d^* - b^* x|}\right)^{* -1} \frac{x - z}{|x - z|} \left(\frac{d^* - b^* z}{|d^* - b^* z|}\right)^{-1} . \end{aligned}$$

For $u \in \Gamma_n$ such that $|u| = 1$, $\tau(u^{*-1}) = \tau(u') = \tau'(u)$ so that

$$\begin{aligned} &\pi'_{-\lambda}(g) J_\lambda f(x) \\ &= \int_{E^n} |x - z|^{2\lambda-n} |d^* - b^* z|^{-2\lambda-n} \tau\left(\frac{d^* - b^* z}{|d^* - b^* z|}\right)^{-1} f((a^* z - c^*)(d^* - b^* z)^{-1}) dz \\ &= J_\lambda \pi_\lambda(g) f(x) . \end{aligned}$$

□

For $s \in \mathbb{C}$, let, for $x \in E^n, x \neq 0$

$$d_s(x) = |x|^{s-1} \sum_{j=1}^n x_j E_j ,$$

and let D_s be the associated convolution operator defined by

$$D_s f(x) = \int_{E^n} d_s(y) f(x - y) dy .$$

Proposition 5.4. *Let $\Re s > -n$. For any $g \in \tilde{G}$,*

$$D_s \circ \pi'_{\frac{s+n}{2}}(g) = \pi'_{-(\frac{s+n}{2})}(g) \circ D_s \quad (11)$$

Proof. By using the trivial intertwining relations (6) and (8), we can transform the intertwining relation (10) to get

$$J_\lambda E_1 \circ \pi'_\lambda(g) = \pi'_{-\lambda}(g) \circ J_\lambda E_1 ,$$

for any $g \in \tilde{G}$. Next, for any $x \in E^n, x \neq 0$

$$|x|^{2\lambda-n-1}(x_1 + x_2 E_1 E_2 + \cdots + x_n E_1 E_n) E_1 = d_{2\lambda-n}(x) .$$

The statement follows, with $\lambda = \frac{s+n}{2}$. □

6 The fundamental identity and its consequences

For $\Re s > -n$, d_s is an integrable function on E^n (with values in $\text{End}(\Sigma)$), hence a (tempered) distribution. We want to meromorphically continue this distribution to \mathbb{C} .

Let D be the Dirac operator on E^n . By definition, it acts on smooth functions on E^n with values in Σ by

$$Df(x) = \sum_{j=1}^n E_j \frac{\partial f}{\partial x_j}(x) .$$

Extend this formula to $\text{End}(\Sigma)$ valued function : if $S(x)$ is such a function, let $DR(x) = \sum_{j=1}^n E_j \frac{\partial R}{\partial x_j}(x)$. Notice that the associated convolution operator satisfies

$$D(R \star f) = DR \star f .$$

In both cases, $D^2 f = -\sum_{j=1}^n \frac{\partial^2 f}{\partial x_j^2}$, which we write as $D^2 = \Delta$, where Δ is the (extension to Σ or $\text{End}(\Sigma)$ -valued functions of the) standard Laplacian on E^n .

Proposition 6.1 (Fundamental identity). *Let $s \in \mathbb{C}, \Re s > -n$. Then, for $x \in E^n, x \neq 0$*

$$D|x|^{s+1} = (s+1)d_s , \quad (12)$$

where both sides are $\text{End}(\Sigma)$ -valued functions.

Proof. It amounts to the formula

$$\frac{\partial}{\partial x_j} |x|^{s+1} = (s+1) x_j |x|^{s-1} .$$

□

As the meromorphic continuation of the distribution $|x|^s$ is well known (see [9]), equation (12) allows the meromorphic continuation of the distribution d_s .

Proposition 6.2. *The distribution $|x|^s$ can be continued meromorphically to \mathbb{C} , with simple poles at $s = -n - 2k$, for $k \in \mathbb{N}$. The residue at $s = -n - 2k$ is given by*

$$\text{Res}(|x|^s, -n - 2k) = c_k \Delta^k \delta ,$$

$$\text{where } c_k = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})} \frac{1}{2^k k! n(n+2) \dots (n+2k-2)} .$$

Proposition 6.3. *The distribution d_s can be meromorphically continued to \mathbb{C} with simple poles at $s = -n - 2k - 1$, $k \in \mathbb{N}$. The residue of d_s at $-n - 2k - 1$ is given by*

$$\text{Res}(d_s, -n - 1 - 2k) = c_k D^{2k+1} \delta .$$

Proof. Let f be a function in $\mathcal{C}_c^\infty(E^n, \Sigma)$. Use (12), recall that the E_j 's are skew Hermitian and integrate by part to get

$$\int_{E^n} d_s(x) f(x) dx = \frac{1}{s+1} \int_{E^n} |x|^{s+1} Df(x) dx .$$

This identity is valid *a priori* for $\Re s > -n$. The right hand side can be extended to a meromorphic function, with poles at $s+1 = -n - 2k$, $k \in \mathbb{N}$. This serves to *define* the left hand side. At $s = -n - 2k - 1$, the residue of the right hand side is $c_k \Delta^k (Df)(0)$. But $D^2 = \Delta$, so that $\Delta^k = D^{2k}$, hence the proposition follows.

□

Theorem 6.1. *For any positive integer k , and any $g \in \tilde{G}$,*

$$D^{2k+1} \circ \pi'_{-k-\frac{1}{2}}(g) = \pi'_{k+\frac{1}{2}}(g) \circ D^{2k+1} \quad (13)$$

Proof. Recall the intertwining relation (11). It is clearly valid for s in \mathbb{C} , provided s is not a pole. But at a pole, say $s = -n - 2k - 1$, we may pass to the limit on both sides of (10), thus obtaining (13).

□

7 The compact picture

The results above may also be realized in the compact picture, i.e. on the sphere, where the induced representation, the Knapp-Stein operators and their residues may also be computed. The main result is an explicit expression of the residues as a polynomial in the Dirac operator of the sphere. Recall from [11] that our induced representation may also be realized in the compact picture, see Chapter VII, (7.3a), and the intertwining operator given as in [11] (7.37) as follows: Set, for f a function on K and $x \in K$

$$J_{\tau, \lambda, w} f(x) = \int_K e^{(-\rho + \lambda) \log a(w^{-1}k)} \tau(m(w^{-1}k)) f(xk) dk .$$

Now we can repeat the arguments from Euclidian space and realize the Knapp-Stein operator as a kernel operator, acting on sections of the spin bundle over S^n , and we may find the residues of this meromorphic family. For this is it convenient to calculate the spectrum of the Knapp-Stein operator, i.e. its eigenvalues on the K -types in the induced representation. Recall the method of spectrum-generating [5], which we can apply in an elementary way to obtain the K -spectrum as in the following result.

When n is odd, the spin representation of \widetilde{M} has highest weight $(\frac{1}{2}, \dots, \frac{1}{2})$ (we use the standard choices of a Cartan subalgebra of \mathfrak{m} and a basis inside). The induced representation space (sections of the spin bundle over S^n) decomposes under the action of \widetilde{K} without multiplicity, and the corresponding highest weights of the \widetilde{K} -types are $(j, \pm) = (j, \frac{1}{2}, \dots, \frac{1}{2}, \pm \frac{1}{2})$ with $j \in \mathbb{N} + \frac{1}{2}$.

When n is even, the spin representation of \widetilde{M} is a sum of two representations, say σ^+ and σ^- , with respective highest weights $(\frac{1}{2}, \dots, \frac{1}{2}, \frac{1}{2})$ and $(\frac{1}{2}, \dots, \frac{1}{2}, -\frac{1}{2})$. Each induced representation $((\pi_\lambda^+, \mathcal{S}_\lambda^+)$ from σ_+ , $(\pi_\lambda^-, \mathcal{S}_\lambda^-)$ from σ_-) decomposes under the action of \widetilde{K} without multiplicity, and the corresponding highest weights of the \widetilde{K} -types are $(j, \frac{1}{2}, \dots, \frac{1}{2})$ with $j \in \mathbb{N} + \frac{1}{2}$.

Proposition 7.1. *Define the spectral functions as in [5] by*

$$Z_{j, \pm}(\lambda) = \pm \frac{\Gamma(\frac{n}{2} + j - \lambda)}{\Gamma(\frac{n}{2} + j + \lambda)}$$

When n is odd, the operator acting on the (j, \pm) \widetilde{K} -type by the scalar $Z_{j, \pm}(\lambda)$ is an intertwining operator between π_λ and $\pi_{-\lambda}$.

When n is even, the operator acting from \mathcal{S}_λ^\pm into \mathcal{S}_λ^\mp on the \widetilde{K} -type $(j, \frac{1}{2}, \dots, \frac{1}{2})$ by the scalar $Z_{j, \pm}(\lambda)$ is an intertwining operator between $\pi_\lambda = \pi_\lambda^+ \oplus \pi_\lambda^-$ and $\pi_{-\lambda} = \pi_{-\lambda}^+ \oplus \pi_{-\lambda}^-$.

See [5], noticing that the parameter r there coincides with $-\lambda$ in our present context. Because of the generic uniqueness of the intertwining operator between π_λ and $\pi_{-\lambda}$, the intertwining operator thus constructed is a multiple (by some meromorphic function of λ) of the one we use in the first part. The poles of the former were at $\lambda = -\frac{1}{2} - k, k \in \mathbb{Z}$. They now correspond to non singular values of the spectral functions, so that the residues are replaced by true values. The normalization is in fact such that the value of the intertwining operator at $\lambda = -\frac{1}{2}$ is exactly the Dirac operator on S^n . More precisely, for $\lambda = -\frac{1}{2}$, we obtain the spectrum of the Dirac operator on S^n .

Proposition 7.2. *The spectrum of the Dirac operator \mathbb{D} is given by*

$$Z_{k+\frac{1}{2}, \pm}(-\frac{1}{2}) = \pm(\frac{n}{2} + k) .$$

Remark. An alternative determination of the spectrum of the Dirac operator on S^n was given in [4], using a more complicated argument which however is only using conformal geometry). See also [6].

For the other poles $\lambda = -\frac{1}{2} - m, m \in \mathbb{N}$, the computation of the values of the spectral functions and the previous result yield the following result.

Theorem 7.1. *Let $m \in \mathbb{N}$. The differential operator*

$$\mathbb{D}_m = \mathbb{D}(\mathbb{D}^2 - 1)(\mathbb{D}^2 - 4) \dots (\mathbb{D}^2 - m^2)$$

is covariant with respect to $(\pi_{-\frac{1}{2}-m}, \pi_{\frac{1}{2}+m})$.

Proof. For $j = \frac{1}{2} + k$,

$$Z_{k+\frac{1}{2}, \pm}(-\frac{1}{2} - m) = \pm(\frac{n}{2} + k + m)(\frac{n}{2} + m - 1) \dots (\frac{n}{2} + k - m)$$

which coincides with the spectral function of the operator

$$\mathbb{D}_m = (\mathbb{D} + m)(\mathbb{D} + m - 1) \dots (\mathbb{D} - m) .$$

Hence the statement. □

For other approaches to the covariance properties of powers of the Dirac operator on S^n , see [8], [12].

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